Compressed sensing: basics and beyond (tutorial)

Compressed Sensing Basics and Beyond

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S.M. Fosson E. Magli

POLITECNICO DI TORINO

Department of Electronics and Telecommunications

CRISP

Towards Compressive Information Processing Systems

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Outline

1. Mathematical problem
2. Applications
3. Recovery
4. Distributed compressed sensing

Outline

Mathematical problem

Compressed sensing (compressed sampling, compressive sensing... CS) deals with

Underdetermined linear systems ...

$$Ax = y$$

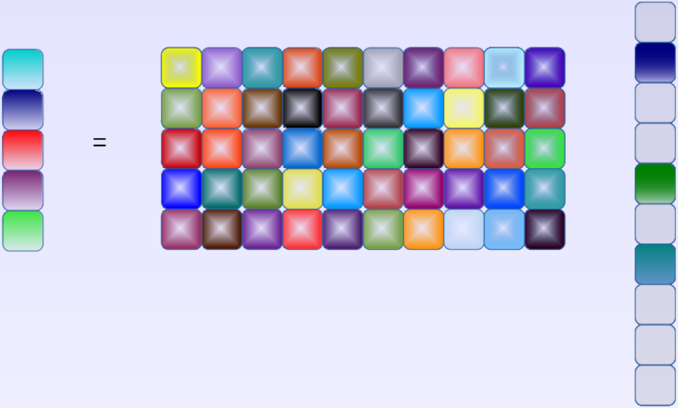
$x \in \mathbb{R}^n$ (unknown), $y \in \mathbb{R}^m$ (measurements), $A \in \mathbb{R}^{m \times n}$, $m < n$

Within the infinite set of solutions, CS looks for the sparsest one

... with sparsity assumptions

x is k-sparse, i.e., it has k non-zero components, where $k \ll n$

Mathematical problem



Questions

$$Ax = y, x \in \mathbb{R}^n(\text{sparse}), y \in \mathbb{R}^m, m < n$$

1. Is the problem well-posed (= is the solution unique)?
2. Are there feasible algorithms to find the solution?
3. Which applications motivate this study?

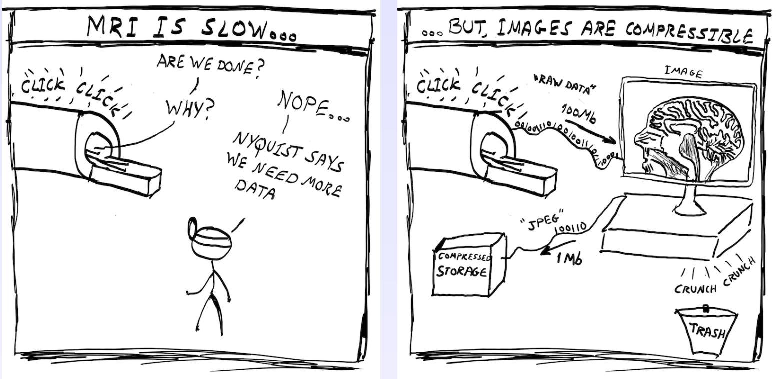
Answers

1. Yes, under some conditions
2. A number of recovery algorithms have been proposed
3.
   - Sparsity is ubiquitous: many signals are sparse in some basis ($y = A\phi x$ where ϕ is the sparsifying basis, e.g., DCT, wavelets, Fourier...)
   - Applications where data acquisition is difficult/expensive, and one aims to move the computational load to the receiver

Outline

Medical Imaging

Magnetic Resonance Imaging (MRI): acquisition is slow [Lustig (2012)]

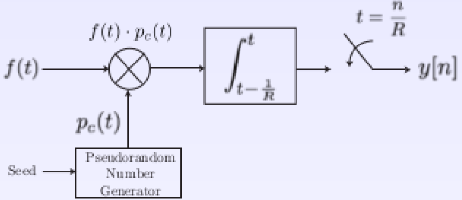


→ sense the compressed information directly

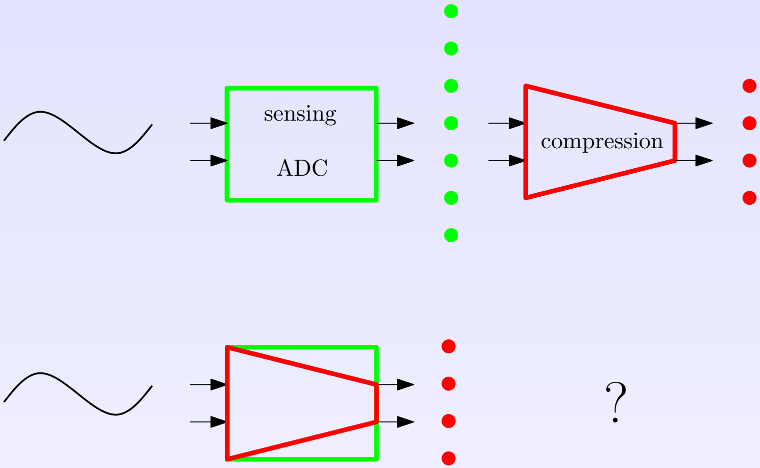
Compression and sampling

$$Ax = y, x \in \mathbb{R}^n(\text{sparse}), y \in \mathbb{R}^m, m < n$$

- Sampling: Nyquist-Shannon sampling theorem states given a signal bandlimited in (B, B), to represent it over a time interval T, we need at least $2BT$ samples
- CS indicates a way to merge compression and sampling, and sample at a sub-Nyquist rate [Tropp et al. (2009)]

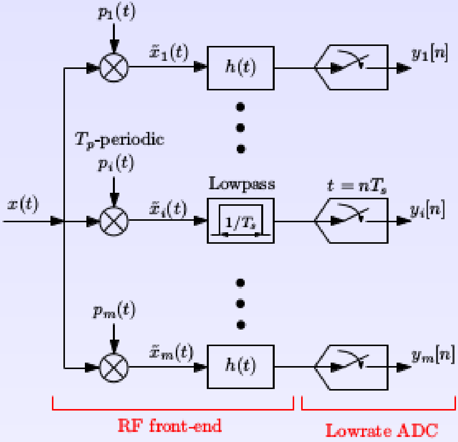


Compression and sampling



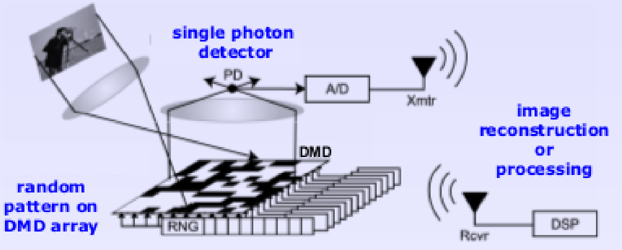
Wideband spectrum sensing

Modulated wideband converter (MWC) [Mishali and Eldar (2010)]



- Sub-Nyquist sampling for signals sparse in the frequency domain
- Realized in hardware (with commercial devices)

Single-pixel camera



Boufonos et al., ICASSP 2008

Key ingredient: a microarray consisting of a large number of small mirrors that can be turned on or off individually

Light from the image is reflected on this microarray and a lens combines all the reflected beams in one sensor, the single pixel of the camera

Outline

Mathematical formulation

ℓ_0-norm

$\|x\|_0 :=$ number of nonzeros entries of $x \in \mathbb{R}^n$

Natural formulation of the CS problem:

$$P_0 : \quad \min_{x \in \mathbb{R}^n} \|x\|_0 \text{ subject to } Ax = y$$

- Is the solution unique?
- P_0 is NP-hard!

Uniqueness of the solution

Spark

spark(A) $:=$ minimum number of columns of A that are linearly dependent

Theorem [D. Donoho, M. Elad (2003)]

For any vector $y \in \mathbb{R}^m$, there exists at most one k-sparse signal $x \in \mathbb{R}^n$ such that $y = Ax$ if and only if spark(A) $> 2k$.

Uniqueness of the solution

Coherence

$\mu(A) := \max_{i \neq j} \frac{|A_i^T A_j|}{\|A_i\|_2 \|A_j\|_2}$ ($A_i = i$th column of A)

Theorem [D. Donoho, M. Elad (2003)]

If

$$k < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right)$$

$y \in \mathbb{R}^m$, there exists at most one k-sparse signal $x \in \mathbb{R}^n$ such that $y = Ax$.

Possible solution: convex relation

Basis Pursuit

$$P_1: \quad \min_{x \in \mathbb{R}^n} \|x\|_1 \text{ subject to } Ax = y$$

- P_1 is convex; can be solved by linear programming
- Are P_0 and P_1 equivalent?

Coherence

Coherence

$\mu(A) := \max_{i \neq j} \frac{|A_i^T A_j|}{\|A_i\|_2 \|A_j\|_2}$ ($A_i = i$th column of A)

Theorem [Elad and Bruckstein (2002)]

If for the sparset solution $x^\star$ we have

$$\|x^\star\|_0 < \frac{\sqrt{2} - \frac{1}{2}}{\mu(A)}$$

then the solution of P_1 is equal to the solution of P_0.

Restricted Isometry Property (RIP)

RIP

Matrix A satisties the RIP of order k if there exists $\delta_k \in (0, 1)$ such that the following relation holds for any k-sparse x:

$$(1 - \delta_k) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k) \|x\|_2^2$$

Theorem [Candès (2008)]

If $\delta_k < \sqrt{2} - 1$, then for all k-sparse $x \in \mathbb{R}^n$ such that $Ax = y$, the solution of P_1 is equal to the solution of P_0.

Which matrices?

- Coherence, spark, RIP: not easy to compute
- Random matrices A with i.i.d. entries drawn from continuous distributions have $\text{spark}(A) = m + 1$ with probability one.
- Gaussian, Bernoulli matrices: given $\delta \in (0, 1)$ there exist c_1, c_2 depending on δ such that G. and B. matrices satisfy the RIP with constant δ and any $m \geq c_1 k \log(n/k)$ with probability $\geq 1 - 2e^{-c_2 m}$ [Baraniuk (2008)]
- Structured matrices: circulant matrices, partial Fourier matrices

Orthogonal Matching Pursuit (OMP)

- "When we talk about BP, we often say that the linear program can be solved in polynomial time with standard scientific software, and we cite books on convex programming [...]. This line of talk is misleading because it may take a long time to solve the linear program, even for signals of moderate length" [Tropp and Gilbert (2007)]
- Possible solution: greedy algorithm, fast, easy to implement $\rightarrow$ OMP

Orthogonal Matching Pursuit (OMP)

1. Initialize $r_0 = y$, $\Lambda_0 = \emptyset$
2. For $t = 1, \dots, T_{max}$
3. $\lambda_t = \underset{j=1,\dots,n}{\operatorname{argmax}} |A_j^T r_{t-1}|$
4. $\Lambda_t = \Lambda_{t-1} \cup \{\lambda_t\}$
5. $\widehat{x}_t = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \|y - A_{\Lambda_t} x\|_2$
6. $r_t = y - A_{\Lambda_t} \widehat{x}_t$

- $T_{max} \approx k$
- OMP requires the knowledge of k!

Variants of BP

Basis Pursuit Denoise (BPDN)

$$P_1 : \quad \min_{x \in \mathbb{R}^n} \|x\|_1 \text{ subject to } \|Ax = y\|_2 \leq \varepsilon$$

Unconstrained version of BPDN

Lasso

$\min_{x \in \mathbb{R}^n} \left(\|Ax - y\|_2^2 + \lambda \|x\|_1 \right)$

For some $\lambda > 0$, Lasso and BPDN have the same solution (the choice of λ is tricky!)

Iterative soft thresholding (IST)

1. $\widehat{x}_0 = 0$
2. For $t = 1, \ldots, T_{max}$
3. $\widehat{x}_t = \mathsf{S}_\lambda(\widehat{x}_{t-1} + \tau A^T(y - A * \widehat{x}_{t-1}))$

where the operator S_λ is defined entry by entry as
$\mathsf{S}_\lambda(x) = \mathrm{sgn}(|x| - \lambda)$ if $|x| > \lambda$, 0 otherwise

- IST achieves a minimum of the Lasso [Fornasier (2010)], and in many common situations such minimum is unique [Tibshirani (2012)]
- Faster method to get a minimum of the Lasso: alternating direction method of multipliers (ADMM)

Iterative hard thresholding

1. $\widehat{x}_0 = 0$
2. For $t = 1, \dots, T_{max}$
3. $\widehat{x}_t = \mathrm{H}_k(\widehat{x}_{t-1} + A^T(y - A\widehat{x}_{t-1}))$

where the operator $\mathrm{H}_k(x)$ is the non-linear operator that sets all but the largest (in magnitude) k elements of x to zero [Blumensath (2008)]

Outline

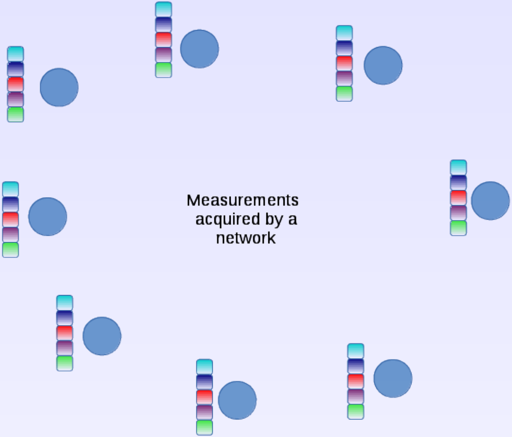
Distributed compressed sensing (DCS)

- Data acquisition is perfomed by a network of sensors

$$y_v = A_v x_v \qquad v \in \mathcal{V} = \{ \text{ sensors } \}$$

- First works: recovery is performed by a fusion center that gathers information from the network (sensing matrices, measurements)
- New: in-network recovery, exploiting local communication and consensus procedures
- We need iterative algorithms

Distributed Compressed Sensing (DCS)



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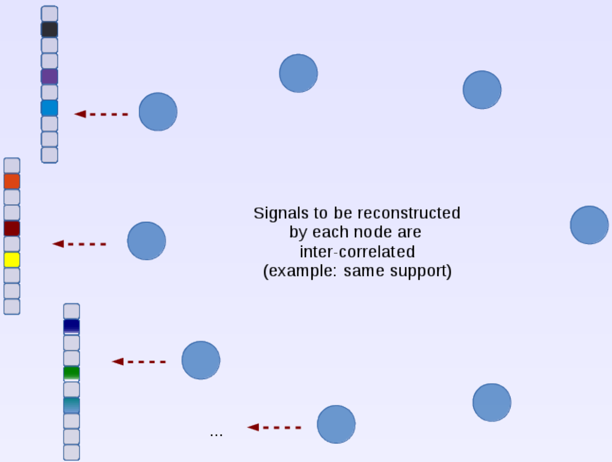

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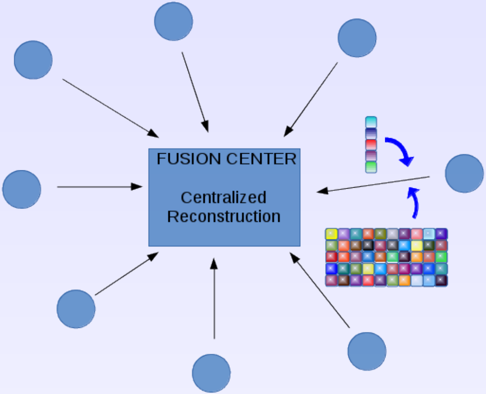
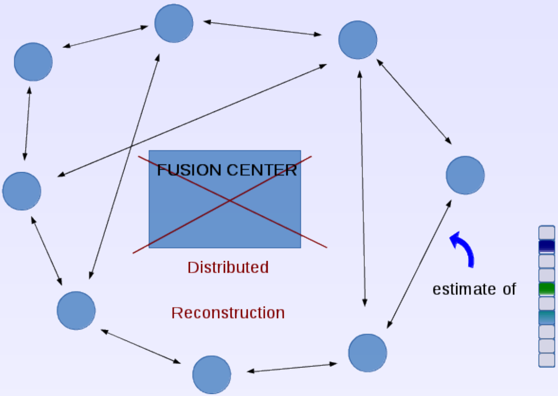


Distributed Compressed Sensing (DCS)



DCS: in-network recovery

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